

Assisted fusion

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A model of nuclear fusion consisting of a wave packet impinging into a well located between square one dimensional barriers is treated analytically. The wave function inside the well is calculated exactly for the assisted tunneling induced by a perturbation mimicking a constant electric field with arbitrary time dependence. Conditions are found for the enhancement of fusion.

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I. INTRODUCTION

Quantum tunneling explains the transmission of particles or clusters of particles through regions that are energetically forbidden. A time-honored success of the quantum tunneling model is the explanation of α decay lifetimes of unstable nuclei by Gurney and Condon and Gamow[1, 2]. More recent revisions of the tunneling paradigm in the nuclear decay context, have reaffirmed its validity.[3, 10].

In previous works we investigated the tunneling of a one dimensional metastable state between barriers, excited by a time dependent potential.[5],[6]

In ref.[6] we found analytical expressions for the assisted tunneling processes. The acceleration of the decay of a metastable state was found to be determined by the poles of the unperturbed wave functions in the complex energy plane.

In the present work we generalize the approach of [6], and find exact solutions to the assisted tunneling of a wave packet impinging from a region outside the range of the potentials, into a square well lying between square barriers. This setup provides a simplified model for the fusion of nuclei.[7]

Fusion time scales are of the order of the transit time of the nuclei past each other and differ from α decay lifetimes markedly. Nuclear decay lifetimes are typically very long in comparison to natural nuclear times. On this basis, it may be argued that fusion can not be affected in the same manner as found for the case of α decay. Nevertheless, in the presence of a long range perturbation, the tunneling process is sensible to longer time scales. Consequently, an external agent can affect the inter penetration the nuclei.

However, for long times, it is not possible to follow the decay process numerically.[4]. Hence, analytical formulas are vital for nuclear fusion as well as for decay. This is the motivation for the present effort.

In the next section we review the model and extend it the case of fusion. In section 3 we apply the results to the penetration of a wave packet into a region between

square barriers. By inspecting the formulas we are able to identify the relevant parameter space for the enhancement of fusion.

II. ANALYSIS OF ASSISTED TUNNELING

The Schrödinger equation for a one dimensional system consisting of a square well between square barriers is [15]

$$i \frac{\partial \Psi}{\partial t} = \frac{-1}{2m} \frac{\partial^2 \Psi}{\partial x^2} - V_0 \Theta(x_0 - |x|) \Psi + \gamma (\Theta(d - |x|) \Theta(|x| - x_0)) \Psi \quad (1)$$

The inclusion of a square well between the barriers is needed to model nuclear fusion.

The even and odd stationary states of eq.(1) for energies below the barrier strength γ are

$$\chi_{e,o}(x) = \frac{\varphi_{e,o}(k)}{\sqrt{\pi n_{e,o}(k)}} \quad (2)$$

where

$$\varphi_e(k) = \begin{cases} \cos(qx); |x| < x_0 \\ A_1 e^{\kappa|x|} + B_1 e^{-\kappa|x|}; d > |x| > x_0 \\ C_1 \cos(kx) + \text{sign}(x) D_1 \sin(kx); |x| > d \end{cases} \quad (3)$$

where κ and q are defined in eq.(26), and $\text{sign}(x) = 1, -1$ for $x > 0, x < 0$.

$$\varphi_o(k) = \begin{cases} \sin(qx); |x| < x_0 \\ \text{sign}(x)(A_2 e^{\kappa|x|} + B_2 e^{-\kappa|x|}); d > |x| > x_0 \\ \text{sign}(x) C_2 \cos(kx) + D_2 \sin(kx); |x| > d \end{cases} \quad (4)$$

, and we have extracted a factor of $\sqrt{\pi}$ from the normalizations for convenience. The labels e, o refer to the even or odd character of the wave functions

The set of even-odd functions is orthonormal and complete.[11, 12] The all important normalization factors of eq.(2), and the amplitudes of eqs.(3,4). are shown in appendix A.

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We subject the system to a time dependent spatially linear perturbation, higher powers in space can be dealt similarly,

$$V(x, t) = \mu x G(t) \quad (5)$$

with μ a coupling constant. For the case of a spatially constant time-harmonic electric field of intensity E_0 interacting with a nucleus of charge $Z|e|$ we have

$$\mu G(t) = Z|e|E_0 \sin(\omega t) \quad (6)$$

Applying a unitary transformation

$$\begin{aligned} \Psi(x, t) &= e^{-i\sigma} \Phi \\ \sigma &= \mu x \zeta + \int \frac{\zeta^2 \mu^2}{2m} dt \\ \zeta &= \int G(t) dt \end{aligned} \quad (7)$$

The Schrödinger equation(1) including the perturbation of eq.(5) becomes

$$\begin{aligned} i \frac{\partial \Phi}{\partial t} &= \frac{-1}{2m} \frac{\partial^2 \Phi}{\partial x^2} + \gamma (\Theta(x_0 + d - |x|)\Theta(|x| - x_0))\Phi \\ &- V_0 \Theta(x_0 - |x|)\Phi + i\zeta(t) \frac{\mu}{m} \frac{\partial \Phi}{\partial x} \end{aligned} \quad (8)$$

Eq.(8) is solved by expanding the wave function in the complete set of even and odd states of the unperturbed Schrödinger equation (2)

$$\Phi(x, t) = \sum_{i=e,o} \int_0^\infty \chi_i(k, x) c_i(k, t) e^{\frac{-i k^2 t}{2m}} dk \quad (9)$$

Taking advantage of the superposition integral evaluated in the appendix A of [6], adapted now to the case of a potential well between the barriers

$$\begin{aligned} I &= \int e^{-\frac{i(k'^2 - k^2)t}{2m}} \chi_e(k, x) \frac{\partial \chi_o(k', x)}{\partial x} dx \\ &= \frac{q}{n_e(k)n_o(k)} \delta(k - k') \end{aligned} \quad (10)$$

with q defined in eq.(26) of Appendix A, the time evolution of the amplitudes $c_{e,o}$ of eq.(9) becomes

$$\begin{aligned} \dot{c}_o(k) &= -\frac{\mu q \zeta}{m n_e(k) n_o(k)} c_e(k) \\ \dot{c}_e(k) &= \frac{\mu q \zeta}{m n_e(k) n_o(k)} c_o(k) \end{aligned} \quad (11)$$

a dot denoting a time derivative.

For an initial ($t = 0$), wave packet representing the incident nucleus carrying mean momentum q_0 and centered at r_0 , $\Psi_{q_0, r_0}(x, t = 0)$, the solution of eq.(11) reads

$$\begin{aligned} c_e(k, t) &= A_e \cos(z) + A_o \sin(z) \\ c_o(k, t) &= A_o \cos(z) - A_e \sin(z) \\ z &= \frac{\mu q Y}{m n_e n_o} \\ Y(t) &= \int_0^t \zeta(t') dt' \end{aligned} \quad (12)$$

where

$$\begin{aligned} A_e &= \frac{\mathbf{A}_e}{n_e} = \int_{-\infty}^\infty \Psi_{q_0, r_0} \chi_e dx \\ A_o &= \frac{\mathbf{A}_o}{n_o} = \int_{-\infty}^\infty \Psi_{q_0, r_0} \chi_o dx \end{aligned} \quad (13)$$

III. ASSISTED TUNNELING INTO A WELL BETWEEN BARRIERS

The most important contribution to eq.(9) in the region of the well is due to the zeros of the normalization factors $n_{e,o}$ of eq.(2) displayed in eqs.(24,25) of appendix A.[5],[6]

Figure 1 shows the inverse of the normalizations of eq.(2) for $m = 15000 \text{ MeV}$, $x_0 = 7 \text{ fm}$, $d = 12 \text{ fm}$, $\gamma = 10 \text{ MeV}$, $V_0 = 40 \text{ MeV}$, corresponding to an ^{16}O nucleus for energies below the barrier. The spikes in figure 1 are due to the extreme closeness of the minima of the normalization factors to their complex zeros.

For eigenenergies above the barrier, the inverse of the amplitudes show a smooth spectrum of peaks of order one in height and width. These peaks will contribute a negligible amount at long times due to the strong oscillations of the time dependent exponential factors of the wave functions, and are henceforth omitted.

The number of poles is directly related to the width and height of the barrier. Very few minima and consequently very few poles show up below the barrier energy for lighter nuclei, as compared to the α decay process from heavier nuclei for which the repulsive barriers are wider and taller.[6]

The normalization factors around their minima can be cast in the form[5]

$$k^2 n_{e,o}^2 \approx \lambda_{j,(e,o)} (k^2 - k_{j,(e,o)}^2)^2 + \beta_{j,(e,o)} \quad (14)$$

where j enumerates the zeros of n_e, n_o .

The poles appear in pairs located symmetrically above and below the real momentum axis[6]. The stationary

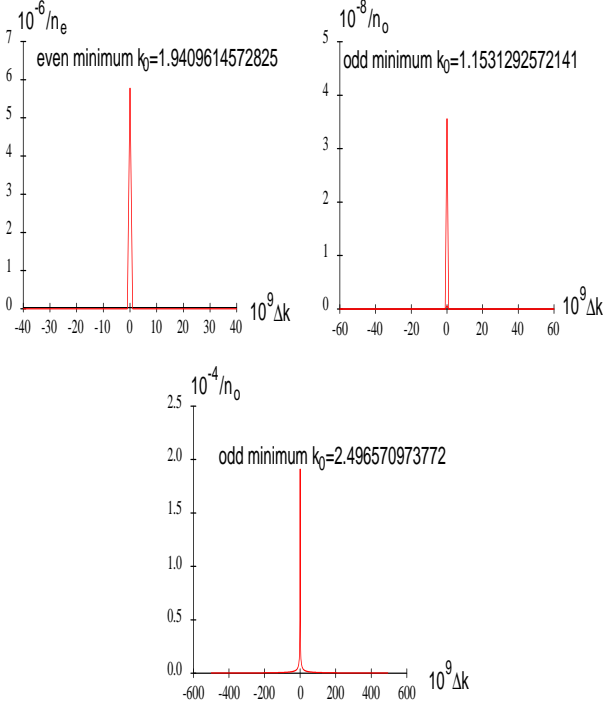


FIG. 1: Normalizations of eq.(2) around the minima at $k = k_0$, with $\Delta k = k - k_0$, *see text*.

wave functions of eqs.(3,4) consist of incoming and outgoing parts that have poles in the upper and lower halves of the complex energy plane respectively[13].

The minima of the normalization factors are extremely sharp. It was necessary to apply a multiple precision numerical package to locate them [8] with a maximum of 50 decimal digits. The imaginary parts of the poles are orders of magnitude smaller than the real parts.[16]

Eq.(9) can be performed now by contour integration in the complex energy plane, closing the contour from below to insure convergence. The negative imaginary part of each pole induces a decaying exponential when the dominant time dependence is due to the phase of the unperturbed wave function. The perturbation can induce both decaying and growing behaviors. In [5],[6] we were interested in the acceleration of the decay. Here, we seek an enhancement of the rate of penetration into the well region, for which a different parameter space will be needed.

After performing the complex energy plane contour integration, the wave function in the region between the barriers becomes

$$\begin{aligned}\Phi &= \Phi_{int} + \Phi_{ext} \\ \Phi_{ext} &= \xi_{1,e} + \xi_{2,o} + \xi_{3,o} + \xi_{4,e} \\ \Phi_{int} &= \sum_{i=e,o} (\Xi_{1,i} + \Xi_{2,i} + \Xi_{3,i} + \Xi_{4,i})\end{aligned}\quad (15)$$

The splitting into an external and internal contribution is necessary because the outer part of the wave function vanishes at a pole. The treatment of the contour integration has to be modified accordingly.

The rather involved expressions of the wave functions of eqs.(15) are spelled out in appendix B.

Using the large argument expansion of the hypergeometric functions[14] in eqs.(29-33) and eqs.(34-37) of Appendix B, the long time $t \rightarrow \infty$ behavior of eq.(15) in the region of the well $|x| < x_0$ becomes

$$\begin{aligned}\Phi(x, t) &\approx \sum_j \frac{C_j(x)}{\sqrt{t}} e^{F_j(t)} \\ F_j(t) &= \frac{\mu k_j^R k_j^I Y(t)}{m |q_0|} - \frac{k_j^R k_j^I t}{m}\end{aligned}\quad (16)$$

where we have used the connection between the even and odd amplitudes at the poles[6]

$$\begin{aligned}n_o(k_{o,j})^2 \beta_{e,j}^2 &= k_{e,j}^2 \\ n_e(k_{e,j})^2 \beta_{o,j}^2 &= k_{o,j}^2\end{aligned}\quad (17)$$

k_j^R, k_j^I denote the real and imaginary parts of the poles, $|q_0| = \sqrt{2 m V_0}$, and $C_j(x)$ is a prefactor that depends on the specific features of the initial wave packet.

Inspection of eq.(16) reveals that the perturbation can either accelerate the tunneling process or slow it down depending on the sign of $F(t)$. This is also true for the assisted tunneling corresponding to α decay. In [5],[6] we focused on assisted decay for which $F(t)$ was demanded to be negative.

It follows from eq.(16) that the perturbation will assist the fusion process when

$$\frac{\mu Y(t)}{|q_0|} > t \quad (18)$$

independently of the pole structure.

To fulfill eq.(18), $Y(t)$ has to be positive definite.

Using eqs.(6,12), $G(t) = \sin(\Omega t)$, with $\Omega \neq 0$,

$$Y(t) = \frac{(\Omega t - \sin(\Omega t))}{\Omega^2} \quad (19)$$

$Y(t) > 0$ is satisfied for $\Omega t \gg 1$.

For long times, eq.(18) becomes

$$\Omega < \frac{\mu}{|q_0|} \quad (20)$$

It is now possible now to estimate the parameters needed for assisted fusion.

Consider low density and temperature totally ionized oxygen nuclei approximated as an ideal gas. For a temperature of $100^0 K$ the velocity of the oxygen nuclei is

$v \approx 400 \frac{m}{sec}$. The inter nuclei distance at $P = 0.1 Pa$ is $\approx 2400 \text{\AA}$. The average time between collisions is $\delta t \approx 6 \cdot 10^{-10} sec$. If we take $\Omega \delta t = 10$, eq.(20) becomes

$$\mu > 10 \frac{|q_0|}{t} \quad (21)$$

Inserting the fusion parameters used in figure 1 into eq.(21) we find

$$\mu > 600 \frac{MeV}{cm} \quad (22)$$

or equivalently an electric field amplitude in eq.(6) of around $|E_0| \approx 8 \cdot 10^7 \frac{Volt}{cm}$.

This electric field appears quite large but not unreachable, especially in light of the fact that the corresponding frequency of eq.(20) is $\Omega \approx 17 GHz$ and the perturbation

can be applied in ultrashort pulses of nanosecond duration.

For angular frequencies smaller than the value prescribed by eq.(20), the tunneling into the well of a packet located far away outside the well, will increase as compared to the unassisted case.

Undoubtedly, the actual fusion problem is more complicated than the simplified model of a packet impinging on a well between barriers, especially because of nuclear structure aspects. However, the analysis of tunneling provided here suggests that the enhancement of fusion by means of external time dependent agents is possible.

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 - [15] In the following our units are *fm* for length and time, and *MeV* or *fm⁻¹* for energies, momenta and mass γ , V_0 , m .
 - [16] The numerical results of [6] were not obtained with the multiple precision package FMLIB. The errors incurred can be estimated to be at most 15%. The product of the even and odd amplitudes at a minimum is now found to be exactly equal to one, and not approximately so.

IV. APPENDIX A: AMPLITUDES OF THE UNPERTURBED WAVE FUNCTIONS

The amplitudes of the unperturbed wave functions of eqs.(3,4) are

$$\begin{aligned} A_1 &= -\frac{1}{2\kappa e_1}(qs_1 - \kappa c_1) \\ B_1 &= -\frac{e_1}{2\kappa}(qs_1 + \kappa c_1) \\ A_2 &= \frac{1}{2\kappa e_1}(qc_1 + \kappa s_1) \\ B_2 &= -\frac{e_1}{2\kappa}(qc_1 - \kappa s_1) \end{aligned} \quad (23)$$

$$\begin{aligned} ((n_e(k))^2 &= (C_1(k)^2 + D_1(k)^2) \\ C_1 &= \frac{1}{2 e_1 e_2 q k} (e_2^2 q \kappa s_1 s_2 - e_2^2 \kappa^2 c_1 s_2 \\ &\quad + e_1^2 \kappa^2 c_1 s_2 + e_1^2 q \kappa s_1 s_2 \\ &\quad - e_2^2 q k s_1 c_2 + e_2^2 \kappa k c_1 c_2 \\ &\quad + e_1^2 \kappa k c_1 c_2 + e_1^2 q k s_1 c_2) \\ D_1 &= \frac{1}{2 e_1 e_2 q k} (-e_2^2 q \kappa s_1 c_2 + e_2^2 \kappa^2 c_1 c_2 \\ &\quad - e_1^2 \kappa^2 c_1 c_2 - e_1^2 q \kappa s_1 c_2 \\ &\quad - e_2^2 q k s_1 s_2 + e_2^2 \kappa k c_1 s_2 \\ &\quad + e_1^2 \kappa k c_1 s_2 + e_1^2 q k s_1 s_2) \end{aligned} \quad (24)$$

$$\begin{aligned} ((n_o(k))^2 &= (C_2(k)^2 + D_2(k)^2) \\ C_2 &= \frac{1}{2 e_1 e_2 q k} (-e_2^2 q \kappa c_1 s_2 - e_2^2 \kappa^2 s_1 s_2 \\ &\quad + e_1^2 \kappa^2 s_1 s_2 - e_1^2 q \kappa c_1 s_2 \\ &\quad + e_2^2 q k c_1 c_2 + e_2^2 \kappa k s_1 c_2 \\ &\quad + e_1^2 \kappa k s_1 c_2 - e_1^2 q k c_1 c_2) \end{aligned}$$

$$\begin{aligned}
D_2 = & \frac{1}{2 e_1 e_2 q k} (e_2^2 q \kappa c_1 c_2 + e_2^2 \kappa^2 s_1 c_2 \\
& - e_1^2 \kappa^2 s_1 c_2 + e_1^2 q \kappa c_1 c_2 \\
& + e_2^2 q k c_1 s_2 + e_2^2 \kappa k s_1 s_2 \\
& + e_1^2 \kappa k s_1 s_2 - e_1^2 q k c_1 s_2)
\end{aligned} \quad (25)$$

$$\begin{aligned}
\kappa &= \sqrt{2 m \gamma - k^2} \\
q &= \sqrt{2 m (k^2 + V_0)} \\
e_1 &= e^{\kappa x_0} \\
e_2 &= e^{\kappa d} \\
c_2 &= \cos(k d) \\
s_2 &= \sin(k d) \\
c_1 &= \cos(q x_0) \\
s_1 &= \sin(q x_0)
\end{aligned} \quad (26)$$

V. APPENDIX B: WAVE FUNCTION IN THE REGION BETWEEN THE BARRIERS

The even and odd amplitudes of eq.(13) are written as

$$\begin{aligned}
A_{e,o} &= \frac{\mathbf{A}_{e,o}}{n_{e,o}} \\
&= \frac{\mathbf{A}_{ext}^{e,o} + \mathbf{A}_{int}^{e,o}}{n_{e,o}} \\
\mathbf{A}_{ext}^{e,o} &= \int_{-\infty}^{\infty} \Theta(|x| - d) \Psi_{q_0, r_0} \chi_{e,o} dx \\
\mathbf{A}_{int}^{e,o} &= \int_{-\infty}^{\infty} \Theta(d - |x|) \Psi_{q_0, r_0} \chi_{e,o} dx
\end{aligned} \quad (27)$$

Splitting of the contributions of the inner and outer regions is needed because the outer part of the wave functions vanishes at the poles. The finite part of the time dependent wave function for the outer regions has to be evaluated by a slightly different method than the contribution of the inner part.

For the sake of exemplification, we consider an initial Gaussian wave packet carrying mean momentum q_0 and centered at r_0 ,

$$\begin{aligned}
\Psi_{q_0, r_0}(x, t = 0) &= e^{\phi} \\
\phi &= i q_0 (x - r_0) - \frac{(x - x_0)^2}{\Delta^2}
\end{aligned} \quad (28)$$

Inserting eq.(27) into the wave function of eqs.(15) for the region between the barriers, the contour integration around the poles yield

$$\Xi_{1,e} = \sum_j \sum_{n=0}^{\infty} Q_{ee} \frac{(-X_{j,e}^2)^n}{(n!)^2}$$

$$\begin{aligned}
Q_{ee} &= \sqrt{\pi} \mathbf{A}_e^{int} \cos(k_{j,e} x) e^{w_e} \\
&\frac{k_{j,e}}{2 \sqrt{2\beta_{j,e} \lambda_{j,e}}} \\
\Xi_{1,o} &= \sum_j \sum_{n=0}^{\infty} Q_{eo} \frac{(-X_{j,o}^2)^n}{(n!)^2 (2n+1)(n+1)} \\
Q_{eo} &= -\sqrt{\pi} \mathbf{A}_o \cos(k_{j,o} x) e^{w_o} \mu^2 Y(t)^2 q_{j,o}^2 \\
&\frac{k_{j,o}}{4 \sqrt{2\beta_{j,o} \lambda_{j,o}} m^2 n_e^4(k_{j,o})}
\end{aligned} \quad (29)$$

where

$$\begin{aligned}
X_{j,(e,o)} &= \mu q_{j,(e,o)} Y(t) \\
&\frac{k_{j,(e,o)}}{2 m n_o(k_{j,(e,o)}) \sqrt{\beta_{j,(e,o)}}} \\
w_{e,o} &= -k_{j,(e,o)}^2 \left(\frac{\Delta^2}{4} + \frac{i t}{2 m} \right) \\
q_{j,(e,o)} &= \sqrt{(k_{j,(e,o)})^2 + 2 m V_0}
\end{aligned} \quad (30)$$

$$\begin{aligned}
\Xi_{2,e} &= \sum_j \sum_{n=0}^{\infty} \tilde{Q}_{ee} \frac{(-X_{j,e}^2)^n}{(n!)^2 (2n+1)} \\
\tilde{Q}_{ee} &= \mu Y(t) q_{j,e} \sqrt{\pi} \mathbf{A}_o \cos(k_{j,e} x) e^{w_e} \\
&\frac{k_{j,e}}{2 m n_o^2(k_{j,e}) \sqrt{2\beta_{j,e} \lambda_{j,e}}} \\
\Xi_{2,o} &= \sum_j \sum_{n=0}^{\infty} \tilde{Q}_{eo} \frac{(-X_{j,o}^2)^n}{(n!)^2 (2n+1)} \\
\tilde{Q}_{eo} &= \mu Y(t) q_{j,o} \sqrt{\pi} \mathbf{A}_o^{int} e^{w_o} \\
&\frac{k_{j,o} \cos(k_{j,o})}{2 m n_e^2(k_{j,o}) \sqrt{2\beta_{j,o} \lambda_{j,o}}}
\end{aligned} \quad (31)$$

$$\begin{aligned}
\Xi_{3,e} &= \sum_j \sum_{n=0}^{\infty} Q_{oe} \frac{(-X_{j,e}^2)^n}{(n!)^2 (2n+1)(n+1)} \\
Q_{oe} &= -\mu^2 Y(t)^2 q_{j,e}^2 \sqrt{\pi} \mathbf{A}_o^{int} \sin(k_{j,e} x) e^{w_e} \\
&\frac{k_{j,e}}{4 \sqrt{2\beta_{j,e} \lambda_{j,e}} m^2 n_o^4(k_{j,e})} \\
\Xi_{3,o} &= \sum_j \sum_{n=0}^{\infty} Q_{oo} \frac{(-X_{j,o}^2)^n}{(n!)^2} \\
Q_{oo} &= \sqrt{\pi} \mathbf{A}_o \sin(k_{j,o} x) e^{w_o} \\
&\frac{k_{j,o}}{2 \sqrt{2\beta_{j,o} \lambda_{j,o}}}
\end{aligned} \quad (32)$$

$$\begin{aligned}
\Xi_{4,e} &= \sum_j \sum_{n=0}^{\infty} \tilde{Q}_{oe} \frac{(-X_{j,e}^2)^n}{(n!)^2 (2n+1)} \\
\tilde{Q}_{oe} &= -\mu Y(t) q_{j,e} \sqrt{\pi} \mathbf{A}_e^{int} \sin(k_{j,e} x) e^{w_e}
\end{aligned}$$

$$\begin{aligned}
\Xi_{4,o} &= \sum_j \sum_{n=0}^{\infty} \tilde{Q}_{oo} \frac{k_{j,e}}{2 m n_o^2(k_{j,e}) \sqrt{2\beta_{j,e}\lambda_{j,e}}} \frac{(-X_{j,o}^2)^n}{(n!)^2 (2n+1)} \\
\tilde{Q}_{oo} &= -\mu Y(t) q_{j,o} \sqrt{\pi} \mathbf{A}_e \sin(k_{j,o} x) e^{w_o} \\
&\frac{k_{j,o}}{2 m n_e^2(k_{j,o}) \sqrt{2\beta_{j,o}\lambda_{j,o}}} \quad (33)
\end{aligned}$$

$$\begin{aligned}
\xi_{1,e} &= \sum_j \sum_{n=0}^{\infty} Q_{ee}^{ext} \frac{(-X_{j,e}^2)^n (4n)!}{(2n!)^3} \\
Q_{ee}^{ext} &= \frac{\sqrt{\pi} \mathbf{A}_e^{ext} \cos(k_{j,e} x) e^{w_e}}{2 \sqrt{\lambda_{j,e}}} \quad (34)
\end{aligned}$$

$$\begin{aligned}
\xi_{2,o} &= \sum_j \sum_{n=0}^{\infty} \tilde{Q}_{eo}^{ext} \frac{(-X_{j,o}^2)^n (4n)!}{(2n!)^2 (2n+1)} \\
\tilde{Q}_{eo}^{ext} &= \frac{\mu Y(t) q_{j,o} \sqrt{\pi} \mathbf{A}_o^{ext} \cos(k_{j,o} x) e^{w_o}}{2 m n_e^2(k_{j,o}) \sqrt{\lambda_{j,o}}} \quad (35)
\end{aligned}$$

$$\begin{aligned}
\xi_{3,o} &= \sum_j \sum_{n=0}^{\infty} Q_{oo}^{ext} \frac{(-X_{j,o}^2)^n (4n)!}{(2n!)^3} \\
Q_{oo}^{ext} &= \frac{\sqrt{\pi} \mathbf{A}_o^{ext} \sin(k_{j,o} x) e^{w_o}}{2 \sqrt{\lambda_{j,o}}} \quad (36)
\end{aligned}$$

$$\begin{aligned}
\xi_{4,e} &= \sum_j \sum_{n=0}^{\infty} \tilde{Q}_{oe}^{ext} \frac{(-X_{j,e}^2)^n (4n)!}{(2n!)^2 (2n+1)} \\
\tilde{Q}_{oe}^{ext} &= -\frac{\mu Y(t) q_{j,e} \sqrt{\pi} \mathbf{A}_{ext,e} \sin(k_{j,e} x) e^{w_e}}{2 m n_o^2(k_{j,e}) \sqrt{\lambda_{j,e}}} \quad (37)
\end{aligned}$$

where $k_{j,e}$ is the complex momentum at the pole number j of the even unperturbed wave function, $Y(t)$ is defined in eq.(12), and λ and β correspond to the expansion around a pole of eq.(14). The wave functions of eqs.(29-33) and eqs.(34-37), can be expressed in terms of standard hypergeometric, Struve, and Bessel functions.[14]